Section 3.1 & 3.2

(Re)Introducing the Derivative

Our motivating example for limits was the instantaneous rate of change. We return to that topic now.

Definition 3.1.1. The *average rate of change* in f on the interval $[a, x]$ is the slope of the secant line through $(a, f(a))$ and $(x, f(x))$ given by

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The *instantaneous rate of change* in f at a is

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is also the *slope of the tangent line* to f at $(a, f(a))$ provided the above limit exists. The *tangent line* is the unique line through $(a, f(a))$ with slope $m_{\tan}$. Its equation is

$$y = m_{\tan}(x - a) + f(a).$$

Example 3.1.2. In Example 2.1.2, we used a table to find the slope of the tangent line to $f(x) = 15 - 2x^2$ at $x = 1$. We can now use limit techniques to find this slope and be confident in our answer.

Alternate Definition 3.1.1. The *average rate of change* in f on the interval $[a, a+h]$ is the slope of the corresponding secant line:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{h}.$$

The *instantaneous rate of change* in f at a is

$$m_{\text{tan}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 3.1.3. Use the alternate definition of i.r.c. to find the equation of the tangent line to $f(x) = \dfrac{1}{x}$ at $x = 2$.

The slope of the line tangent to the graph of a function is such useful information that it has a special name:

Definition 3.1.4 (The Limit Definition of Derivative)**.** The *derivative of f at a*, denote $f'(a)$, is given by either of the two limits, provided that they exist and provided a is in the domain of f:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, we say f is *differentiable* at a.

Example 3.1.5. Let $g(x) = \dfrac{1}{x^2}$. Compute $g'(2)$ and use it to find the slope of the tangent line at $(2, g(2))$.

Example 3.1.6. Sound intensity I, measured in watts per square meter at a point x meters from a sound source with acoustic power P, is given by $I(x) = \frac{P}{4\pi x^2}$. A sound system at a rock concert produces an acoustic power around $P = 3\,W$. Compute and interpret $I'(2)$.

The Derivative as a Function

The derivative of f at a measures the slope of the line tangent to f at the point $(a, f(a))$. As a changes so does the tangent line change and expectedly so too does its slope. We can then consider the derivative as a function where we let a be our independent variable.

Definition 3.2.7 (The Derivative Function)**.** The *derivative* of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 3.2.8. Let $f(x) = \sqrt{x}$. Compute $f'(x)$.

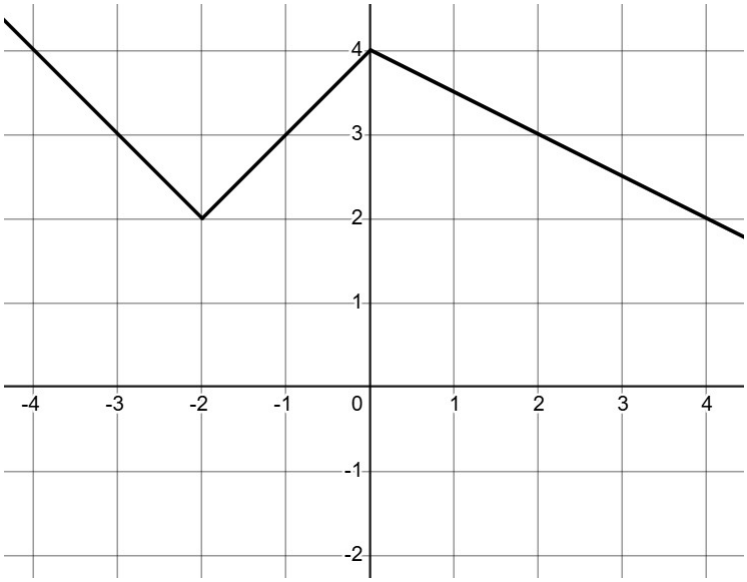
Notation 3.2.9.

Other notations for $f'(x)$ are $\frac{dy}{dx}, \frac{df}{dx}, \frac{d}{dx}\big(f(x)\big), D_x\big(f(x)\big)$. We may change the variable and function names such as $g'(t) = \frac{dg}{dt}$. If we wish to evaluate the derivative of f at the point $x = a$ we write

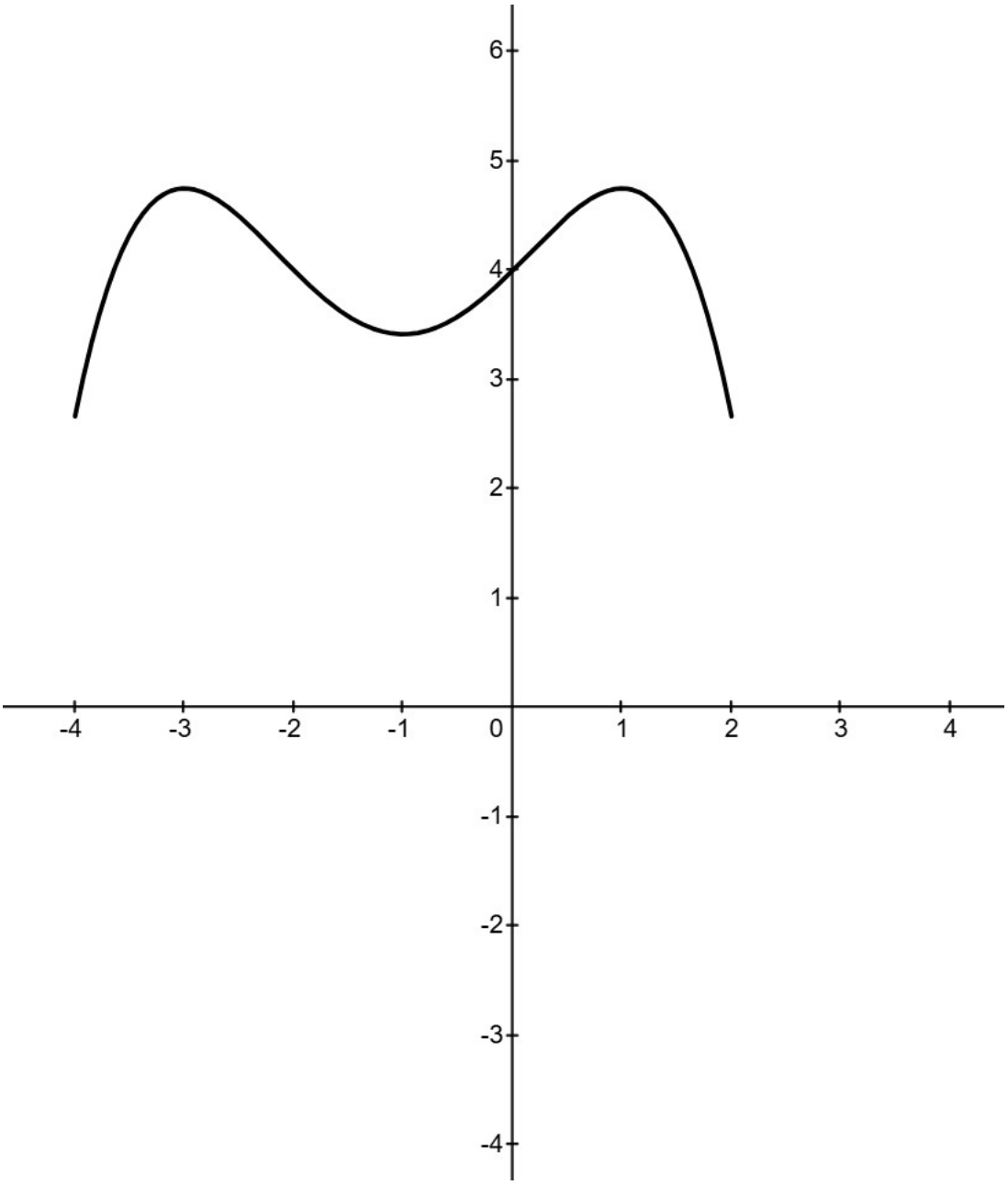
$$f'(a) \text{ or } \left.\frac{df}{dx}\right|_{x=a} \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

Example 3.2.10. Let $g(t) = \dfrac{1}{t^2}$. Compute $\dfrac{dy}{dt}$ and $\left.\dfrac{dy}{dt}\right|_{t=2}$.

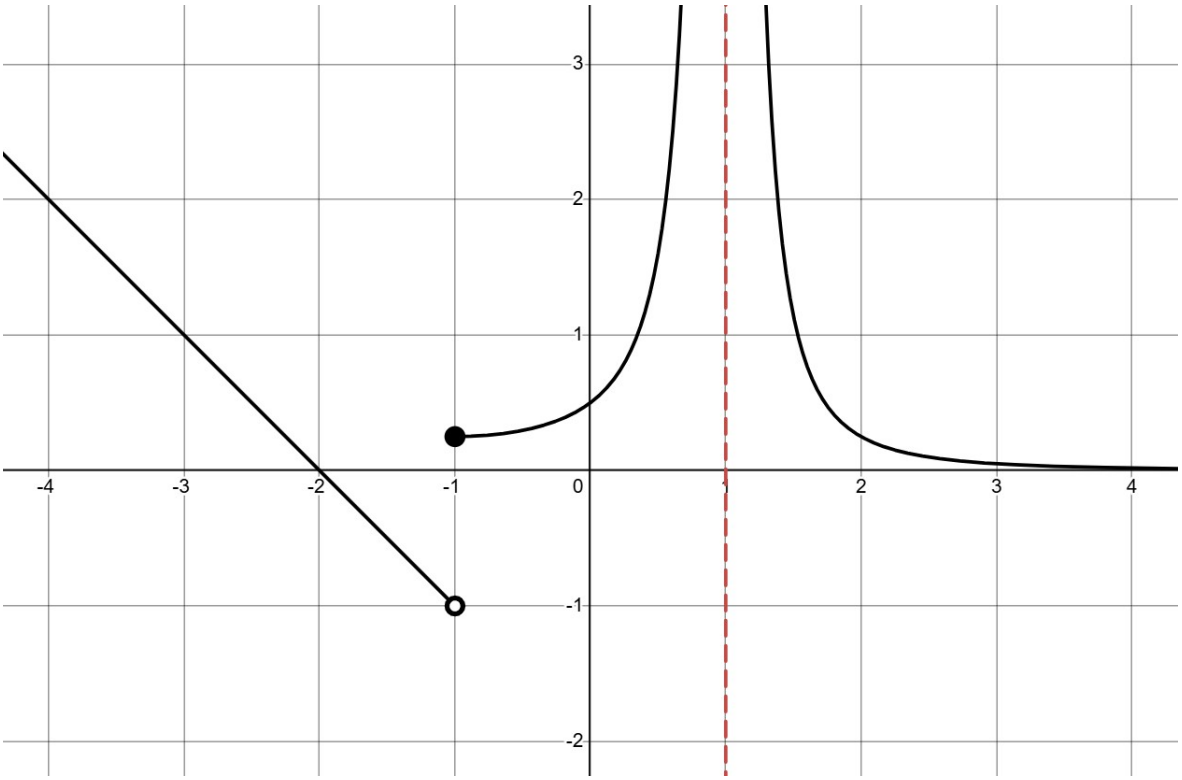
Example 3.2.11. Sketch the graph of f' using the graph of f shown below.



Example 3.2.12. Sketch the derivative of g.



Example 3.2.13. Sketch the derivative of h.



Theorem 3.2.14. If f is differentiable at a, then f is continuous at a.

Alternatively, if f is not continuous at a, then f is not differentiable at a.

Proof:

Non-differentiability. A function f is *not* differentiable at a if at least one of the following are true:

1. f is not continuous at a.
2. f has a corner at a.
3. f has a vertical tangent at a.